Sailing Flight of Large Birds over Land.

THE explanation of this, in NATURE (p. 518), seems inadequate.

If a large body of air be moving uniformly both in respect of direction and of velocity, no matter at what rate, it might as well be perfectly motionless, in respect of its ability to aid the flight of a bird that is simply floating in it. But in fact the air never is motionless: it moves with the earth, from west to east, at the rate, let us assume, of 500 miles an hour. To a bird floating in the air, whether the earth beneath it moves exactly as the air moves, or not, must be a matter of perfect indifference. The earth's relative motion does not affect it. I must myself adhere to the explanation which I gave in a former number of NATURE (vol. xxviii. p. 28), that the birds avail themselves of differences of the movement of the air, in respect of velocity, or of direction, or of both. Mr. S. E. Peal has noticed that "flocks which drift over the hills recover their position on the plains by descending to windward." This is simple enough. The wind is flowing from the plains towards the hills. It rises then as it flows, and has many inequalities in its direction and rate. On entering a gorge a narrow current of air would be thrown upwards with very rapid ascent. Of all the inequalities the birds know how to avail themselves. R. COURTENAY.

Tean Vicarage, September 28.

A Remarkable Meteor.

ON Sunday, September 29, at 7.30 p.m., I observed a very brilliant meteor falling nearly perpendicular a little to the west of north. Its progress towards the earth appeared to be much slower than is usually the case with such bodies, the heavens being illuminated for several seconds. The meteor was of a bright sapphire hue; preceding it were a few drops of bright fiery red, whilst following it came a brilliant trail of light. It seems to have been pretty generally observed throughout Ireland, and letters to the Press from counties Roscommon, Galway, Kilkenny, and Kildare, testify to the interest it has awakened in the country. RICHARD CLARK.

113 Upper Leeson Street, Dublin, October 7.

THE METHOD OF QUARTER SQUARES.[1]

THE method of quarter squares consists in the use of the formula

$$ab = \tfrac{1}{4}(a+b)^2 - \tfrac{1}{4}(a-b)^2$$

to effect the multiplication of two numbers, a and b. If we are provided with a table giving the values of $\frac{1}{4}n^2$ up to a given value of n, we may obtain, by the aid of this formula, without performing any multiplication, the product of any two numbers whose sum does not exceed the limit of the table.

The method is specially interesting on account of the great simplicity of the formula, by means of which a table of double entry may be replaced by one of single entry. How great a transformation is effected by such a change is evident, if we consider that the largest existing multiplication table of double entry reaches only to 1000 × 1000, and forms a closely-printed folio of 900 pages, but that a table of quarter squares of the same extent (*i.e.* of $\frac{1}{4}n^2$ up to $n = 2000$) need only occupy 4 octavo pages. The disparity becomes even more conspicuous as the limit of the table is extended, for a table of double entry extending to 10,000 × 10,000, would require nearly 100 folio volumes; and one extending to 100,000 × 100,000, would require nearly 10,000 volumes; whereas the corresponding quarter-square tables need only occupy 40 and 400 octavo pages respectively.

The use of a table of squares in effecting multiplications was recognized as far back as 1690, when Ludolff published his large table of squares, extending to 100,000. In the introduction to the table Ludolff explained how it could be employed in multiplications. In order to multiply a and b the table is to be entered with $a+b$ and $a-b$ as arguments, and the difference of the corresponding squares divided by 4. If a and b are both even, or both uneven, their sum and difference will both be even numbers, so that $\frac{1}{2}(a+b)$ and $\frac{1}{2}(a-b)$ will be integers. In either of these two cases we may therefore enter the table with the semi-sum and semi-difference of the numbers as arguments, the product being the simple difference of the corresponding squares.

It was not, however, till 1817 that a table of *quarter* squares (*i.e.* of $\frac{1}{4}n^2$ for argument n) was published, for the purpose of facilitating multiplications. If n be uneven, $\frac{1}{4}n^2$ consists of an integer and the fraction $\frac{1}{4}$. This fraction $\frac{1}{4}$ may be ignored in the use of the table, for if either $a+b$ or $a-b$ is uneven, the other is so too; the fraction $\frac{1}{4}$ therefore occurs in both squares, and disappears from their difference. It may therefore be omitted from the table.

The table of 1817, which contained the first practical application of the method, was published by Antoine Voisin, at Paris, under the title "Tables des Multiplications; ou, Logarithmes des Nombres entiers depuis 1 jusqu'à 20,000." It is curious that Voisin should have called a quarter square a logarithm: he called a the root, and $\frac{1}{4}a^2$ its logarithm. His table extended to 20,000, and was thus available for multiplications up to 10,000 × 10,000. On the title-page Voisin described it as effecting multiplications up to 20,000 by 20,000. This statement is justified by the formula

$$ab = 2\{\tfrac{1}{4}a^2 + \tfrac{1}{4}b^2 - \tfrac{1}{4}(a-b)^2\},$$

by which the product was to be obtained if the sum of the numbers exceeded 20,000, the method of quarter squares being then no longer available. It is to be observed, however, that this formula requires three entries besides the final duplication.

Almost simultaneously (1817) a similar table, of the same extent, was published independently by A. P. Bürger at Carlsruhe. The method was rediscovered by J. J. Centnerschwer, who published a table of the same extent in 1825 at Berlin. In 1832, J. M. Merpaut published, at Vannes, a table of quarter squares extending to 40,000. In 1852, Kulik (well known for his large table of squares and cubes to 100,000), who had again rediscovered the method, published a table extending to 30,000. In 1856, Mr. S. L. Laundy published, at London, the largest table of quarter squares which had appeared previous to the publication of the present table. Laundy's table extends to 100,000. It was intended that the multiplications should be effected by means of quarter squares if the sum of the numbers did not exceed 100,000, but other five-figure numbers were to be multiplied by means of Voisin's three-entry formula referred to above.

It is this change of method that has detracted so greatly from the value of Laundy's fine table. It is evident that the table should have been carried to double its actual extent, *i.e.* to 200,000, so that any two five-figure numbers could be multiplied together by means of the two-entry formula. The late General Shortrede constructed such a table, but it was never printed. In the work under notice Mr. Blater carries the table as far as 200,000; so that, more than sixty years after the publication of the first table effecting the multiplication of two four-figure numbers, the extension to five figures has at last been completed.

The method of quarter squares has had no opportunity of a fair trial in the absence of a table extending to 200,000. Considering the many purposes to which Crelle's tables (which give the product of any two three-figure numbers by a single entry) are continually applied, it is perhaps surprising that no general use should ever have been made of a table which in a very small compass gives, by only two entries, the product of two four-figure numbers. Still it is clear that the full power of the method is not felt till we are provided with such a table giving the product of two five-figure numbers. As already stated, the fact that the limit of Laundy's table was only 100,000 has deprived it of most of its value, for it is obvious that, unless all five-figure numbers can be treated by a uniform method, the table could not be conveniently employed in practice.

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[1] "Table of Quarter Squares of all Whole Numbers from 1 to 200,000 for simplifying Multiplication, Squaring, and Extraction of the Square Root, and to render the Results of these Operations more certain." Calculated and published by Joseph Blater. (London: Trübner and Co., 1888.)

Mr. Blater's work consists of the principal table (giving quarter squares up to 200,000), which occupies 200 pages; a small table of four pages, called the index, to facilitate the use of the table in the extraction of square roots; and an introduction, &c., of fourteen pages.

The arrangement of the table (in which the author has followed the plan adopted by Kulik in his table of 1852, already referred to) is somewhat peculiar. The table is first entered (*i.e.* the required page of the table is found) by means of the *last* three figures of the number: the table is then entered on this page (or, more correctly, double page), by means of the preceding figures. For example, the quarter square corresponding to 126,993 is found by turning to the double page headed 990. In one of the four columns headed 993 we enter the table at the line 126: from this line we obtain, in the first column, the first four figures of the result, 4031; in the column under 993, the next three, 805; from the bottom of the column we take the last three figures, 512. The result is therefore given in three parts A, B, C; A being common to ten numbers (in the same line) beginning with 126, C being common to fifty numbers (in the same column) ending with 993, and B being special to the combination 126,993.

The table is beautifully printed in large antique figures on thick and excellent paper, and is a handsome piece of typography. The author mentions that it was entirely set up by a single compositor at the printing-office of Mr. Falk, at Mayence, and that it occupied his whole time for eleven months. Besides being admirably printed, the table is no doubt very correct, as a triple calculation was made, and no pains seem to have been spared by Mr. Blater for insuring accuracy.

The book is dedicated to Mr. Anthony Steinhauser, of Vienna, who has contributed a short historical preface. Mr. Steinhauser, who is himself the author of several logarithmic tables, encouraged Mr. Blater in his work, and rendered him great assistance throughout. The actual calculation occupied eighteen months.

With respect to the general employment of Mr. Blater's table for the performance of multiplications, it is to be feared that its utility has been jeopardized by the size of page adopted. Anyone who has had occasion to make constant use of tables knows the enormous advantage of the octavo form over the quarto. The book is placed to the left of the computer, and the effort of carrying by the eye a series of figures from the left-hand page of a quarto table to the paper—a distance of 18 inches to 2 feet—is but ill compensated for by larger figures or fewer pages. Handsome as the book is to look at, it seems to us that the table would have had much more chance of bringing the method into general use if it had been of octavo form. It is greatly to be regretted that it was not printed on 400 octavo instead of 200 quarto pages, which would have been quite possible with the existing arrangement of the table. If this had been done, and if the type had been somewhat smaller, a neat and handy volume might have been produced.

The mode of entering the table is very insufficiently explained in the introduction. This is unfortunate, as the mode of entry (by the last figures) is so unusual in tables that it should have been explicitly mentioned. Also the translation into English is so very unsatisfactory as to be obscure in places. These, however, are minor blemishes which would have but slight influence on the general utility of the table, if only the form were convenient.

The question of how far the method of quarter squares is likely to come into use is of some interest. Hitherto the method has been very little known, and, so far as we know, it has never been used in practice on any extended scale. The mere fact that it has been so constantly discovered anew is sufficient evidence of the slight attention that it has received. Still, there ought to be room for a table that gives, to the last figure, the products of any two five-figure numbers with only two entries. A seven-figure table of logarithms is inadequate for this purpose, for, besides requiring three entries, it only gives the first seven figures of the result. On the other hand, it may be said that in ordinary calculations seven figures are as many as are required, and that logarithms possess the advantage of being equally convenient for divisions and multiplications. It must be admitted that a five-figure quarter-square table is appropriate to only a limited class of calculations: it applies only to multiplications, and the number of figures in each of the two numbers must not be greater than five. These conditions are of a somewhat special kind. In recent years when heavy multiplications have been required it has become the custom to make use of Thomas de Colmar's arithmometer; and probably, at the present time, nearly all systematic work of this character is carried out either by Crelle's tables or by the arithmometer.

Passing now to the general question of multiplication by means of a table of single entry, we have the two methods of quarter squares and logarithms, each possessing its special advantages. There is also an older method which passed out of notice with the invention of logarithms. This method was called "prosthaphæresis," and depended on the formula

$$\sin a \sin b = \tfrac{1}{2}[\sin\{90^\circ - (a - b)\} - \sin\{90^\circ - (a + b)\}].$$

A table of natural sines could therefore be used as a multiplication table, four entries being required. This method is due to Wittich, of Breslau, who was assistant for a short time to Tycho Brahé, and it was used by them in their calculations in 1582. It is referred to by Raymarus Ursus, Clavius, and Longomontanus; and it seems to have been used for performing multiplications not only when the numbers occurred as sines but also in the case of ordinary numbers.

The method of quarter squares depends upon so simple a formula, that it is strange that the first table should not have appeared until 1817. There seems no reason why it should not have been employed before the invention of logarithms, when it would have been a most valuable aid to calculation. The geometrical theorem, which is equivalent to the algebraical identity $(a + b)^2 - (a - b)^2 = 4ab$, on which the method depends, forms Prop. viii. of the second book of Euclid; and one would think that the application of the geometrical or algebraical theorem to arithmetic might have been noticed at any time. The actual history of mathematical tables is, however, entirely different from what we might expect it to have been, owing to the wonderfully early invention of logarithms: and it was, in fact, only just about that time that the importance of tables as an aid to general calculation was beginning to be felt. The date of Herwart ab Hohenburg's great double-entry multiplication table, extending to 1000 × 1000 (the same limit as Crelle's table, and which has never been exceeded) is only four years earlier (1610) than that of Napier's "Canon Mirificus" (1614).

It is interesting to notice that the method of quarter squares is more closely connected mathematically with the method of prosthaphæresis than with that of logarithms; in fact, from the formula

$$\sin a \sin b = \tfrac{1}{2}\{\cos(a - b) - \cos(a + b)\},$$

we readily deduce

$$ab = \tfrac{1}{4}\{(a + b)^2 - (a - b)^2\},$$

by expanding the sines and cosines in ascending powers of their arguments and equating the terms of two dimensions.

The method of quarter squares enables us to multiply together two numbers of n figures each if we have a table extending to 2×10^n. If the latter only extends to 10^n three entries are required, and the final result has to be doubled whenever the sum of the numbers exceeds 10_n (as in Laundy's table). If we consider the question of the multiplication of two numbers of n figures each by means of a table extending only to 10^n, the same process being employed in all cases, it appears that three entries are necessary, and that it would be better to tabulate half squares, using the formula

$$ab = \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 - \tfrac{1}{2}(a-b)^2,$$

In tabulating the half squares the fraction $\frac{1}{2}$ would be thrown off, so that if a and b were both uneven, unity would have to be added to the result.

It would, however, we think, if the table is not to go beyond 10^n, be more convenient to employ a table of triangular numbers. The nth triangular number is $\frac{1}{2}n(n \times 1)$, and if we are provided with a table extending to 10^n we may multiply any two numbers not exceeding 10^n by means of the formula

$$ab = \tfrac{1}{2}(a-1)a + \tfrac{1}{2}b(b+1) - \tfrac{1}{2}(a-b-1)(a-b)\,;$$

or, as we may write it

$$ab = \mathrm{T}(a-1) + \mathrm{T}(b) - \mathrm{T}(a-1-b),$$

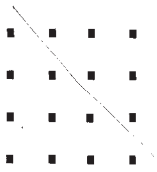
$\mathrm{T}(n)$ denoting the nth triangular number.[1]

Thus, to multiply two numbers we subtract unity from the larger number, and enter the table with the larger number so diminished, with the smaller number, and with the difference of these two numbers. For example, to multiply 5289 and 2156, we add the tabular results corresponding to 5288 and 2156, and subtract from this sum the tabular result corresponding to 3132.

The mode of construction of a table of triangular numbers is almost the simplest possible, the numbers being formed by adding to zero the natural numbers 1, 2, 3, ... *e.g.*,

$$0+1=1,\ 1+2=3,\ 3+3=6,\ 6+4=10,\ 10+5=15,$$

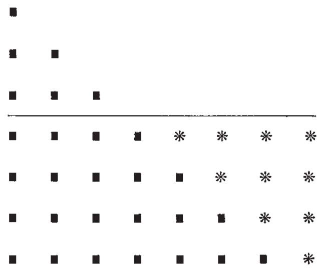
and so on. It may be noticed also that any two consecutive triangular numbers are the most nearly equal parts into which a square of points can be divided by a line parallel to the diagonal. For example, in the square of 16 points, the two most nearly equal triangular parts are, $1+2+3=6$, and $1+2+3+4=10$. This is shown in the following diagram :—



Whether the square contains an even or an uneven number of points, the diagonal, which is in the middle, has to be given to one of the two parts, which therefore necessarily differ by the number of points it contains. In the square n^2, the two consecutive triangular numbers which form it are $\frac{1}{2}n(n-1)$ and $\frac{1}{2}n(n+1)$, differing, as they should, by n, the number of points in the diagonal. Viewing the same matter from a slightly different point of view, we see that any two consecutive triangular numbers always make a square, *e.g.*,

$$1+3=4,\quad 3+6=9,\quad 6+10=16,\quad \&c.$$

It is interesting to exhibit by means of a diagram the manner in which the rectangle representing the product ab is derived from the three triangular numbers corresponding to $a-1$, b, $a-1-b$. As an example, the mode of formation of the product 8×4 is shown below, the triangular number corresponding to 7 being represented by dots and the triangular number corresponding to 4 by stars :—



The dots above the line form the triangular number corresponding to $7-4=3$.[1]

It is not suggested that the method just described by means of triangular numbers is comparable to that of quarter squares. It is certainly better to double the extent of the table and have but two entries. Still, it is interesting to note how readily a table of triangular numbers extending only to 10^n is available for the performance of multiplications of two n-figure numbers. So far as we know, only one extended table of triangular numbers has ever been published. This table, which gives the value of $\frac{1}{2}n(n+1)$ from $n=1$ to $n=20{,}000$, was published at the Hague, by E. de Joncourt, in 1762, under the title "De Natura et Præclaro Usu Simplicissimæ Speciei Numerorum Trigonalium." The book is a small and handsomely printed volume of 267 pages, 224 of which are occupied by the table.

In tabulating quarter squares, the fraction $\frac{1}{4}$ which occurs when the square is uneven is omitted. If we denote by qsq n the tabulated quarter square of n, we have, therefore—

$$\begin{aligned}\text{qsq}\,(2n) &= n^2,\\ \text{qsq}\,(2n+1) &= n^2+n.\end{aligned}$$

A table of quarter squares may be formed by adding to zero the numbers 1, 1, 2, 2, 3, 3, ... *e.g.* $0+1=1$, $1+1=2$, $2+2=4$, $4+2=6$, $6+3=9$, $9+3=12$, and so on. Its construction, therefore, is

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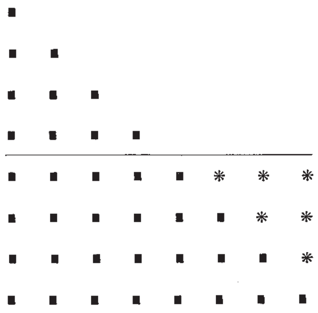
[1] It is interesting to compare the two formulæ which involve half squares and triangular numbers respectively. In the former case we tabulate a discontinuous function, and in the use of the formula a unit has sometimes to be arbitrarily added. In the latter case we tabulate a continuous function, and the formula always holds good (the larger of the arguments being always reduced by unity). One formula depends on squares, n^2; the other on factorials of the second order, $n(n-1)$.

[1] We might of course also perform the multiplication thus :—



corresponding to the formula

$$ab = \mathrm{T}(a) + \mathrm{T}(b-1) - \mathrm{T}(a-b).$$

But if unity is subtracted from the smaller, instead of from the larger, number, slightly higher numbers are involved in the process.

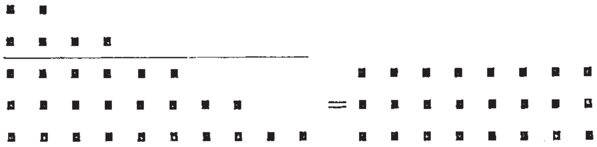
very similar to that of a table of triangular numbers, the only difference being that the added numbers 1, 2, 3, ... are each twice repeated. We may also regard the tabulated quarter squares as defined by this rule: The quarter square of n is equal, if n be even, to the sum of all the uneven numbers less than n, and, if n be uneven, to the sum of all the even numbers less than n. For evidently the series $1 + 3 + 5 + \ldots + (2n - 1) = n^2$, and the series $2 + 4 + 6 + \ldots + 2n = n^2 + n$.

By means of this definition of a quarter square we may exhibit the method of quarter squares diagrammatically as follows.

Taking as examples the products 8 × 3 and 7 × 4, we have

$$\text{qsq } 11 - \text{qsq } 5 = 8 \times 3,$$

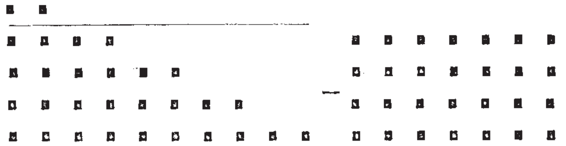
which may be represented by—



and

$$\text{qsq } 11 - \text{qsq } 3 = 7 \times 4,$$

which may be represented by—



The number of points in the extreme left-hand column of the difference of the quarter squares is always equal to the smaller of the numbers to be multiplied. If this number is uneven, there will be one middle line containing a number of points equal to the greater of the two numbers; the points in excess of this number are to be transferred from the line below the middle one to the line next above it, the excess from two lines below is to be transferred to make up the deficiency two lines above, and so on. If the smaller number is even, as in the second diagram, there are two middle lines differing from each other by two points; one point from the lower of these lines is to be transferred to the upper, three points from the line below the lower middle line to the line above the upper middle line, and so on.

It will be noticed that the tabulated quarter squares are, as it were, a species of triangular number in which the succeeding lines of points differ by two, instead of by one, as in ordinary triangular numbers (*i.e.* viewing the matter arithmetically, the quarter squares are derived alternately from the series $1 + 3 + 5 + \ldots$ and $2 + 4 + 6 + \ldots$, and the triangular numbers from the series $1 + 2 + 3 + \ldots$). It is the fact of the lines differing by two which enables us in all cases to adjust the points in the difference of two quarter squares so as to form a rectangle in the manner indicated above.

J. W. L. GLAISHER.

NOTES.

THE American Philosophical Society at Philadelphia is about to celebrate the centennial anniversary of the first occupation of its present hall. The celebration will be held on November 21. The Hon. Frederick Fraley, the President of the Society, will deliver an address in the afternoon of that day, and in the evening a banquet will be given at the Bellevue. The Society was founded in 1743 "for promoting useful knowledge."

ON November 16, the Naturwissenschaftlicher Verein of Bremen will celebrate the twenty-fifth anniversary of its foundation by an evening gathering and a banquet.

THE International Medical Congress will meet next year in Berlin, from August 4 to 10. Inquiries by intending visitors should be addressed to the General Secretary, Dr. Lassar, Karl Strasse, Berlin. The Congress will be divided into eighteen Sections, and the official languages will be German, English, and French.

THE Congress of the International Geodesic Association was opened in Paris, at the Foreign Office, on October 3. M. Spuller welcomed the delegates, who represented Austria, Belgium, Denmark, Spain, France, Greece, Hamburg, Hesse-Darmstadt, Italy, Holland, Prussia, Roumania, Servia, and Switzerland. M. Faye presided.

THE Ethnographic Congress, which held meetings of its various Sections every day last week in Paris, brought its proceedings to a close on Monday afternoon in one of the large halls of the College of France. It was decided that the Congress should hold its next meeting at Bucharest in the autumn of 1890.

THE next annual meeting of the Mineralogical Society will be held in the apartments of the Geological Society, Burlington House, London, on Tuesday, November 5, at 8 p.m.

THE Committee of the Sunday Lecture Society have decided that twenty-one lectures shall be given, during the winter, in St. George's Hall, Langham Place, on Sunday afternoons at 4 p.m., as in former years. The first lecture, on "The Origin and Uses of the Colours of Animals," with oxy-hydrogen lantern illustrations, will be delivered by Dr. Alfred Russel Wallace on October 20. This will be followed by lectures by Mr. John M. Robertson, Mr. Arthur Nicols, Mr. Chas. Cassal, Dr. Andrew Wilson, Prof. Percy Frankland, and Sir R. S. Ball.

ON Monday afternoon, about a quarter to 2 o'clock, a shock of earthquake was felt in Cornwall. It was accompanied by a loud underground noise like thunder, and was felt distinctly at Doubleborough, Boscastle, and Camelford. The earthquake is said to have been severe enough to shake houses, but no harm was done either to person or property.

THE death is announced from Georgetown, British Guiana, of Mr. E. E. H. Francis, Professor of Chemistry in Queen's College, Georgetown, and Analytical Chemist to the Government, at the early age of thirty-nine. Mr. Francis entered the service of British Guiana in 1875, having formerly been Professor of Chemistry and official analytical chemist in Trinidad. The two posts now vacant in Georgetown are said to be worth more than £700 per annum. Mr. Francis was a member of various learned Societies in this country.

AT the recent meeting of the Iron and Steel Institute, Mr. Frederick Siemens made some remarks during the discussion of Sir Lowthian Bell's paper on gaseous fuel, of which we gave an abstract last week. We draw attention to these remarks on account of the important influence Mr. Siemens's experiments and inventions have had on the application of gas for the heating of furnaces. Mr. Samson Fox had stated that he did not propose to use water gas alone for furnace purposes, but mixed with producer gas or other carburetted gas. Mr. Siemens said that a gas would thus be produced similar to that formed in the Siemens gas producer, which he had applied for a considerable time to furnace purposes. He had lately been experimenting with various gases for furnace use, and had come to the conclusion to divide gases generally into two classes, luminous and non-luminous. The latter, the class to which